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On the hidden symmetry of a one-dimensional hydrogen atom

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Abstract. The Fock method is applied to the problem of a one-dimensional hydrogen atom. Integral Fock equations are obtained in discrete and continuous spectra; the case of zero energy is studied and wavefunctions and normalisation constants are calculated in the momentum representation.

1. Introduction

The problem of a one-dimensional hydrogen atom (1H), i.e. a quantum system with the Hamiltonian $\hat{\mathscr{H}} = -\frac{1}{2}\mathrm{d}^2/\mathrm{d}x^2 - 1/|x|$, originated from the study of the behaviour of a hydrogen atom in a strong magnetic field. A rigorous analysis of 1H in the coordinate representation was performed by Loudon (1959). He proved two specific properties: (i) the 1H has no normal state with a finite energy, i.e. a fall-off to the centre; and (ii) the discrete spectrum of 1H is doubly degenerate. The latter property contradicts a conventional idea of non-degeneracy of the discrete spectrum in one-dimensional motion (Landau and Lifshitz 1974). A mathematical reason for the spectrum being doubly degenerate is that the wavefunctions should have zeros at the singularity of potential. Here we shall demonstrate that this degeneracy can also be explained by the mechanism of hidden O(2) symmetry in the discrete spectrum. This conclusion is drawn on the basis of a method that was developed by Fock (1935) for the hydrogen atom and promoted further development of the theory of quantum systems with hidden symmetry (Englefield 1972).

2. Momentum representation

Stationary states of 1H are described by the Schrödinger equation ($\hbar = \mu = e = 1$):

$$\Psi''(x) + 2(E + 1/|x|)\psi = 0. \tag{1}$$

We multiply equation (1) by $|x|$ and make the Fourier transformation

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}px} a(p)\, \mathrm{d}p.$$

Then, taking into account the formula given by Gelfand and Shilov (1958)

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |x|\, \mathrm{e}^{-\mathrm{i}qx}\, \mathrm{d}x = -\frac{1}{\pi} \mathrm{PV} \frac{1}{q^2}$$

in which PV stands for the principal value, we arrive at the integral equation (integral means in the sense of principal value)

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{p'^2-2E}{(p'-p)^2}a(p')\,\mathrm{d}p'=-2a(p). \tag{2}$$

3. Fock formalism in the discrete spectrum

Consider the region of discrete spectrum ($E<0$). Replace $-2E$ by p_0^2 and rewrite equation (2) as

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{p'^2+p_0^2}{(p'-p)^2}a(p')\,\mathrm{d}p'=-2a(p). \tag{3}$$

Introduce an artificial two-dimensional space with cartesian coordinates (η, ξ) and consider in it a circle with radius p_0 at the origin of coordinates (see figure 1). From figure 1 it is seen that

$$p=p_0\tan\varphi/2. \tag{4}$$

Mapping (4) maps the axis η onto a circle and is called a stereographic projection. Applying (4), equation (3) may be written in the form

$$\frac{p_0}{2\pi}\int_0^{2\pi}\frac{\psi(\varphi')}{1-\cos(\varphi'-\varphi)}\,\mathrm{d}\varphi'=-\psi(\varphi) \tag{5}$$

where the notation

$$\psi(\varphi)=(p^2+p_0^2)a(p)=\frac{p_0^2}{\cos^2\varphi/2}a(p) \tag{6}$$

has been introduced. Now let us analyse equation (5). We shall start with a formal expansion

$$\frac{1}{1-\cos\gamma}=\sum_{m=-\infty}^{\infty}c_m\,\mathrm{e}^{\mathrm{i}m\gamma}. \tag{7}$$

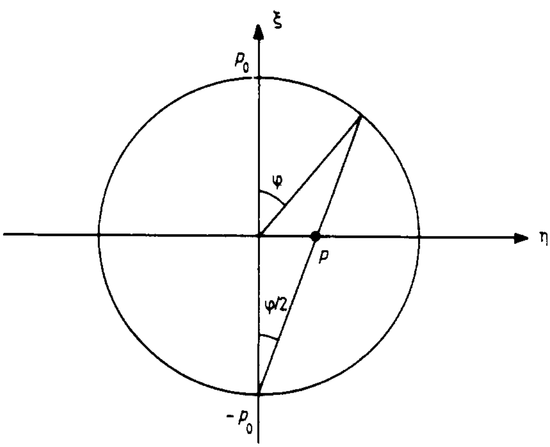

Figure 1. Stereographic projection in the discrete spectrum.

Multiplying (7) by $e^{-im'\gamma}(1-\cos\gamma)$ and integrating over $d\varphi$ in the limits $(0, 2\pi)$ one may derive a three-term recurrence relation

$$2\delta_{m'0} = 2c_{m'} - c_{m'-1} - c_{m'+1}$$

that with the condition $c_{-m} = c_m$ leads to the formula

$$c_m = c_0 - |m|$$

where

$$c_0 = \int_0^{2\pi} \frac{d\gamma}{1-\cos\gamma}.$$

The latter integral is divergent. Nevertheless, using a formal procedure and with the identity (Gelfand and Shilov 1958)

$$\sum_{m=-\infty}^{\infty} e^{im\gamma} = 2\pi \sum_{m=-\infty}^{\infty} \delta(\gamma - 2\pi m)$$

we obtain

$$\frac{1}{1-\cos\gamma} = c_0 2\pi \sum_{m=-\infty}^{\infty} \delta(\gamma - 2\pi m) - \sum_{m=-\infty}^{\infty} |m|\, e^{im\gamma}.$$

Since further integration in (5) is carried out in the sense of principal value, then $\gamma \neq 2\pi m$ and the term containing a divergent constant c_0 gives no contribution, i.e.

$$\mathrm{PV}\frac{1}{1-\cos\gamma} = -\sum_{m=-\infty}^{\infty} |m|\, e^{im\gamma}.$$

By using this formula it may be easily shown that equation (5) has a non-trivial solution only when $p_0 = 1/|m|$, which leads to the following discrete spectrum of 1H found in Landau and Lifshitz (1974):

$$E_m = -1/2m^2 \qquad m = 0, \pm 1, \pm 2, \ldots. \tag{8}$$

According to this formula, fall-off onto the centre and a double degeneration do indeed take place. The functions $\psi(\varphi)$ are of the form

$$\psi_m^{(\pm)}(\varphi) = c\, e^{\pm im\varphi} \qquad m = 1, 2, 3, \ldots. \tag{9}$$

Equation (5) is invariant under the shift $\varphi \to \varphi + \varphi_0$, $\varphi' \to \varphi' + \varphi_0$, which testifies to the hidden 1H symmetry, characteristic of O(2).

4. Calculation of the normalisation constant in the discrete spectrum

The constant c in (9) is determined from the normalisation condition

$$\mathcal{I}_{p_0'p_0}^{(\pm)} = \int_{-\infty}^{\infty} a_{p_0'}^{*(\pm)}(p)\, a_{p_0}^{(\pm)}(p)\, dp = \delta_{p_0 p_0'}. \tag{10}$$

From formulae (6) and (9) it follows that

$$a_{p_0}^{(\pm)}(p) = c\frac{\exp[\pm(2i/p_0)\tan^{-1}(p/p_0)]}{p^2+p_0^2}.$$

Considering that

$$\frac{1}{(p^2+p_0^2)(p^2+p_0'^2)} = \frac{1}{(p_0'^2-p_0^2)}\left(\frac{1}{p^2+p_0^2} - \frac{1}{p^2+p_0'^2}\right)$$

and

$$\frac{\exp[\pm(2i/p_0)\tan^{-1}(p/p_0)]}{p^2+p_0^2}=\pm\frac{1}{2i}\frac{d}{dp}\exp[\pm(2i/p_0)\tan^{-1}(p/p_0)]$$

we arrive at the formula

$$\mathscr{I}^{(\pm)}_{p_0'p_0}=\pm\frac{c(p_0')c(p_0)}{2i(p_0'^2-p_0^2)}\sin\left(\frac{\pi}{p_0}-\frac{\pi}{p_0'}\right)$$

which, with condition (10), gives

$$c=(2/\pi)^{1/2}(m)^{-3/2}.$$

5. Fock formalism in the continuous spectrum

Let us now discuss the continuous spectrum ($E>0$). We introduce the notation $p_0^2=2E$ and rewrite equation (2) in the form

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{p'^2-p_0^2}{(p'-p)^2}a(p')\,dp'=-2a(p). \tag{11}$$

Consider in the two-dimensional space (η, ξ) the hyperbola

$$\xi^2-\eta^2=p_0^2 \qquad \xi=\pm(p_0^2+\eta^2)^{1/2}. \tag{12}$$

There are two regimes of the stereographic projection, depending on whether $p\in(-p_0, p_0)$ or $p\in]-\infty, -p_0; p_0, \infty[$. The first of the regions will be denoted by D_{in} and the second by D_{out}. For $p\in D_{in}$ the regime of projection is shown in figure 2. The

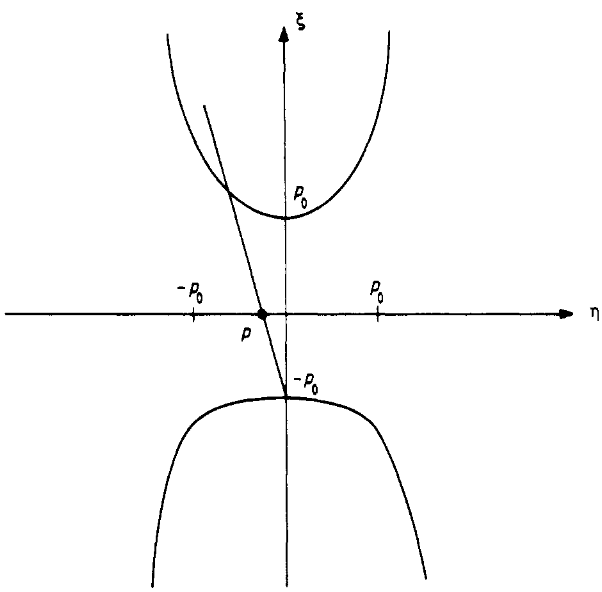

Figure 2. Mapping of the range of momentum p onto the upper branch of the hyperbola $\xi-\eta^2=p_0^2$.

region D_{in} is mapped into an upper branch of the hyperbola (12). From figure 2 it is seen that despite the sign of p the relation

$$p = p_0\eta/(\xi + p_0) \qquad p \in D_{\text{in}} \tag{13}$$

holds. It is convenient to pass to the variable μ $(-\infty < \mu < \infty)$, i.e. to write the equation of the hyperbola upper branch in the parametric form

$$\eta = p_0 \sinh \mu \qquad \xi = p_0 \cosh \mu.$$

Then the mapping (13) acquires the compact form

$$p = p_0 \tanh\tfrac{1}{2}\mu \qquad p \in D_{\text{in}}. \tag{14}$$

The region D_{out} transforms into a lower branch of the hyperbola. In this case the stereographic projection is shown in figure 3 and it leads to the connection

$$p = p_0\eta/(\xi + p_0) \qquad p \in D_{\text{out}}. \tag{15}$$

In formula (15) $\xi < -p_0$, whereas in (13) $\xi > p_0$. Formula (15) is conveniently parametrised by $(-\infty < \mu < \infty)$:

$$\eta = p_0 \sinh \mu \qquad \xi = -p_0 \cosh \mu.$$

In terms of μ, the hyperbola lower branch is defined by the equation

$$p = -p_0 \coth \tfrac{1}{2}\mu \qquad p \in D_{\text{out}}. \tag{16}$$

By using mappings (14) and (15) it can be shown that equation (11) splits into the

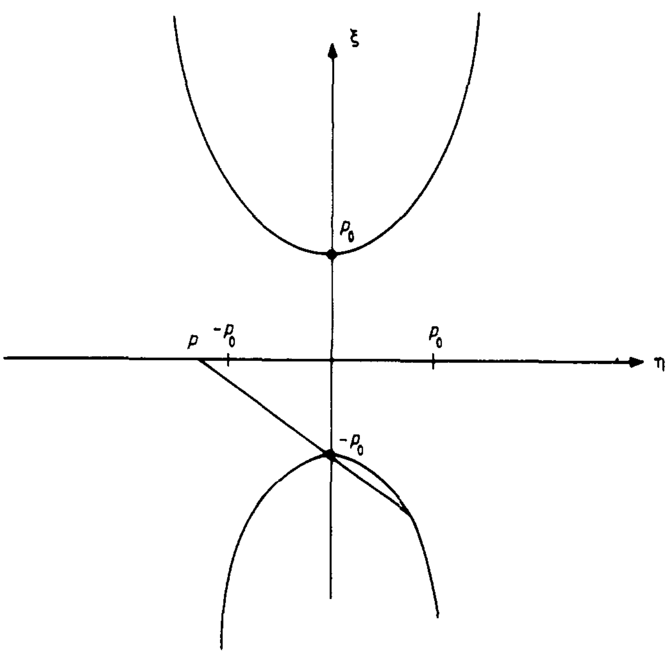

Figure 3. Mapping of the range of momentum p onto the lower branch of the hyperbola $\xi - \eta^2 = p_0^2$.

system of two integral equations

$$\int_{-\infty}^{\infty} \frac{A(\mu')\,d\mu'}{\sinh^2[(\mu'-\mu)/2]} - \int_{-\infty}^{\infty} \frac{B(\mu')\,d\mu'}{\cosh^2[(\mu'-\mu)/2]} = \frac{4\pi}{p_0} A(\mu) \qquad (17)$$

$$\int_{-\infty}^{\infty} \frac{A(\mu')\,d\mu'}{\cosh^2[(\mu'-\mu)/2]} - \int_{-\infty}^{\infty} \frac{B(\mu')\,d\mu'}{\sinh^2[(\mu'-\mu)/2]} = \frac{4\pi}{p_0} B(\mu) \qquad (18)$$

where

$$A(\mu) = \frac{a(p_0 \tanh\frac{1}{2}\mu)}{\cosh^2\frac{1}{2}\mu} \qquad B(\mu) = \frac{a(-p_0 \coth\frac{1}{2}\mu)}{\sinh^2\frac{1}{2}\mu}. \qquad (19)$$

Equations (17) and (18) are invariant under translations $\mu' \to \mu' + \mu_0$ and $\mu \to \mu + \mu_0$, which shows the existence of a hidden symmetry group. Now we proceed to solve the system (17) and (18). We shall need the two integrals

$$\int_{-\infty}^{\infty} \frac{e^{iq\mu}\,d\mu}{\cosh^2\mu} = \frac{\pi q}{\sinh\frac{1}{2}\pi q} \qquad (20)$$

$$\int_{-\infty}^{\infty} \frac{e^{iq\mu}\,d\mu}{\sinh^2\mu} = -\pi q \coth\tfrac{1}{2}\pi q. \qquad (21)$$

The first is taken from tables by Gradshtein and Ryzhyk (1963) and the second is calculated by the formula (Prudnikov *et al* 1981)

$$\int_{-\infty}^{\infty} \frac{e^{-ibz}\,dz}{\sinh z + \sinh a} = -\frac{i\pi\, e^{iab}(\cosh \pi b - e^{-2iab})}{\sinh \pi b \cosh a}.$$

Now we expand the functions $A(\mu)$ and $B(\mu)$ into the Fourier integrals:

$$A(\mu) = \int_{-\infty}^{\infty} \alpha(\tau)\, e^{i\tau\mu}\, d\tau \qquad B(\mu) = \int_{-\infty}^{\infty} \beta(\tau)\, e^{i\tau\mu}\, dt.$$

Inserting these expansions into equations (17) and (18) and using formulae (20) and (21) we arrive at a system of homogeneous equations from which we conclude that τ may assume only two values, $\tau = \pm 1/p_0$, and the functions $A(\mu)$ and $\beta(\mu)$ take the form

$$A_{p_0}^{(\pm)}(\mu) = c\, e^{\pm i\mu/p_0}$$

$$B_{p_0}^{(\pm)}(\mu) = -c\, e^{\pi/p_0}\, e^{\mp i\mu/p_0}.$$

Passing to the initial wavefunctions $a(p)$ by formula (19) we obtain

$$a^{(\pm)}(p_0 \tanh\tfrac{1}{2}\mu) = c \cosh^2\tfrac{1}{2}\mu\, e^{\pm i\mu/p_0}$$

$$a^{(\pm)}(-p_0 \coth\tfrac{1}{2}\mu) = -c \sinh^2\tfrac{1}{2}\mu\, e^{\pi/p_0}\, e^{\mp i\mu/p_0}.$$

These functions can be expressed through the variable p:

$$a_{p_0}^{(\pm)}(p) = \begin{cases} cp_0^2/(p_0^2 - p^2)\exp[\pm(2i/p_0)\tanh^{-1}(p/p_0)] & |p| < p_0 \\ cp_0^2/(p_0^2 - p^2)\exp(\pi/p_0)\exp[\pm(2i/p_0)\tanh^{-1}(p/p_0)] & |p| > p_0. \end{cases}$$

At the points $p = \pm p_0$ the functions $a_{p_0}^{(\pm)}(p)$ cannot be determined.

6. Calculation of the normalisation constant in the continuous spectrum

For definiteness we assume that $p_0' > p_0$ and split the region of p into five subregions: $E_1 = (-\infty, -p_0')$, $E_2 = (-p_0', -p_0)$, $E_3 = (-p_0, p_0)$, $E_4 = (p_0, p_0')$, $E_5 = (p_0', \infty)$. In the region E_3

$$\int_{E_3} a_{p_0'}^{*(\pm)}(p) a_{p_0}^{(\pm)}(p)\,\mathrm{d}p = \lim_{\lambda \to p_0} \int_{-\lambda}^{\lambda} a_{p_0'}^{*(\pm)}(p) a_{p_0}^{(\pm)}\,\mathrm{d}p.$$

In $a_{p_0'}^{(\mp)}(p)$, replacing $\tanh^{-1}(p/p_0')$ by $\tanh^{-1}(p/p_0)$ and separating the difference $p_0 - p_0'$, we obtain

$$\int_{E_3} a_{p_0'}^{*(\pm)}(p) a_{p_0}^{(\pm)}(p)\,\mathrm{d}p = \frac{\pi c^2 p_0^3}{2}\,\delta(p_0' - p_0)$$

and

$$\int_{E_2+E_4} a_{p_0'}^{*(\pm)}(p) a_{p_0}^{(\pm)}(p)\,\mathrm{d}p = \lim_{\lambda_1 \to -p_0'} \lim_{\lambda_2 \to -p_0} \left(\int_{\lambda_1}^{\lambda_2} + \int_{-\lambda_2}^{\lambda_1} \right) a_{p_0'}^{*(\pm)}(p) a_{p_0}^{(\pm)}(p)\,\mathrm{d}p.$$

Considering that at $p_0' \approx p_0$, $\tanh^{-1}(p/p_0') \simeq \tanh^{-1}(p/p_0)$, it can easily be shown that the latter integral vanishes. Finally,

$$\int_{E_1+E_5} a_{p_0'}^{*(\pm)}(p) a_{p_0}^{(\pm)}(p)\,\mathrm{d}p = \tfrac{1}{2}\pi c^2 p_0^3\,\mathrm{e}^{2\pi/p_0}\delta(p_0 - p_0').$$

As a result, we have

$$\int_{-\infty}^{\infty} a_{p_0'}^{*(\pm)}(p) a_{p_0}^{(\pm)}(p)\,\mathrm{d}p = \tfrac{1}{2}\pi c^2 p_0^3(1 + \mathrm{e}^{2\pi/p_0})\delta(p_0 - p_0').$$

Taking the normalisation condition

$$\int_{-\infty}^{\infty} a_{p_0'}^{*(\pm)}(p) a_{p_0}^{(\pm)}(p)\,\mathrm{d}p = 2\pi\delta(p_0 - p_0')$$

we obtain the following normalisation constant:

$$c = \frac{1}{p_0^{3/2}(1 + \mathrm{e}^{2\pi/p_0})^{1/2}}.$$

7. The case $E = 0$

In this case equation (2) becomes

$$\frac{1}{\pi} \fint_{-\infty}^{\infty} \frac{p'^2}{(p' - p)^2}\, a(p')\,\mathrm{d}p' = -2a(p).$$

Here, instead of the stereographic projection, we make the change $\xi = 2/p$ and arrive at the equation

$$\fint_{-\infty}^{\infty} \frac{\psi(\xi')\,\mathrm{d}\xi'}{(\xi' - \xi)^2} = -\tfrac{1}{2}\pi\psi(\xi)$$

where

$$\psi(\xi) = a(p)/\xi^2.$$

From the formula (Gelfand and Shilov 1958)

$$\int_{-\infty}^{\infty} \frac{e^{i\sigma x}}{x^2} dx = -\pi|\sigma|$$

it follows that equation (22) has the solutions

$$\psi^{(\pm)}(\xi) = c\, e^{\pm i\xi}$$

and, consequently,

$$a_{p_0}^{(\pm)}(p) = c\, e^{\pm 2i/p}/p^2.$$

8. Conclusion

The main results are as follows: the Schrödinger equation is obtained for a one-dimensional (1H) hydrogen atom in the momentum representation, the stereographic projection is analysed for $E<0$ and $E>0$, the Fock equations are found for 1H, hidden symmetry is revealed, wavefunctions in the momentum space are found, the method of calculating normalisation constants is shown and a special case $E=0$ is examined. This indicates that a one-dimensional hydrogen atom can be described by the known scheme (Popov 1967, Bander and Itzykson 1966, Shibuya and Wolfman 1965) of the theory of hidden symmetry of hydrogen systems of dimensionality $n \geqslant 2$. The only specific feature is the absence of the ground state with a finite energy.

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